(by $\ell(I+x)=\ell(I)$ $\forall$ interval I & $\forall x\in\mathbb{R}$)

我们已学过 $m^*: 2^{\mathbb{R}} \to [0,+\infty]$ 是个 tran-inv, countably sub add. & that finite sets, countable sets are in $\mathfrak{M}_0$ where $\mathfrak{M}_0$ denotes the family of all A with $m^*(A)=0$. Also

$$m^*(I)=\ell(I) \quad \forall I\in\mathcal{I} \quad \text{(Ex.)}$$

定义 $E\in\mathfrak{M}$ (measurable E) if

$$(*) \quad m^*(A) = m^*(A\cap E) + m^*(A\cap \tilde{E}) \quad \forall A\subseteq 2^{\mathbb{R}}$$

equivalently (by subadditivity of m^* & noting $A=(A\cap E)\cup(A\cap\tilde{E})$)

$$(**) \quad \text{LHS of } (*) \geqslant \text{RHS} \quad \forall A\subseteq 2^{\mathbb{R}} \text{ with } m^*(A)<+\infty$$

Then we will establish:

Th. I. $\mathfrak{M}$ is a tran-inv σ-algebra and $m:=m^*|_{\mathfrak{M}}$ is a measure such that $\mathfrak{M}_0, \mathcal{B}\subseteq\mathfrak{M}$, $m(I)=\ell(I)$ $\forall I\in\mathcal{I}$

proof of Th I. The tran-inv of $\mathcal{M}$ follows from the corresponding property of m^*: $\forall A \subseteq \mathbb{R}$ $\forall x \in \mathbb{R}$, $\forall E \in \mathcal{M}$ one has

$$\begin{cases} m^*(A \cap (E+x)) = m^*((A-x) \cap E) \\ m^*(A \cap \widetilde{(E+x)}) = m^*(A \cap (\tilde{E}+x)) = m^*((A-x) \cap \tilde{E}) \end{cases}$$

Since $E \in \mathcal{M}$, RHS of these two lines are equal so LHS of the two lines are equal, implying $E+x \in \mathcal{M}$. We already showed last week that $\mathcal{M}$ is an algebra, and $\mathcal{M}_0 \subseteq \mathcal{M}$ (indeed, if $m^*(Z)=0$ then, $\forall A \subseteq \mathbb{R}$ with $m^*(A) \leq +\infty$ one has

$$\cancel{m^*(A \cap Z)} + m^*(A \cap \tilde{Z}) = m^*(A \cap \tilde{Z}) \leq m^*(A) \text{ by } \uparrow$$

(zero as $m^* \uparrow$ & $m^*(E)=0$),

implying that $Z \in \mathcal{M}$)

Next to show that $\mathcal{I} \subseteq \mathcal{M}$, we need only show that

$$I := (a, +\infty) \in \mathfrak{M} \quad \forall a \in \mathbb{R}$$

(why?). The essential technique is based on the following device:

Any set A is divided into two parts

$$A' := (-\infty, a) \cap A$$
$$A'' := (a, \infty) \cap A$$

together with possibly $\{a\}$.

Now, $\forall A \subseteq \mathbb{R}$ with $m^*(A) < +\infty$ $\forall \varepsilon > 0$, $\exists$ COIC $\{I_n : n \in \mathbb{N}\}$ of A s.t.

$$m^*(A) + \varepsilon > \sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \ell(I_n') + \ell(I_n'')$$
$$\geq m^*(A') + m^*(A'')$$
$$= m^*\big(A \cap (-\infty, a)\big) + m^*\big(A \cap (a, \infty)\big)$$
$$= m^*\big(A \cap (-\infty, a]\big) + m^*\big(A \cap (a, \infty)\big)$$
$$= m^*(A \cap \tilde{I}) + m^*(A \cap I),$$

implying that $I \in \mathfrak{M}$ as $\varepsilon > 0$ arbitrary.

Therefore $\mathcal{I} \subseteq \mathcal{M}$. Provided that we can show that $\mathcal{M}$ is indeed an σ-algebra, one then $\tau \subseteq \mathcal{M}$ by the Structure Theorem for open sets (any open set G can be represented in the form $G = \bigcup_{n=1}^{\infty} I_n$ with countable pairwise disjoint open intervals I_n. your Exercise to show this). To complete the proof of I, we need a series of lemmas. The 1st lemma below extends the definition of measurability of $E \in \mathcal{M}$ to $E_1 := E$ & $E_2 := \tilde{E}$

Lemma 1. Let $E_1, E_2, \cdots, E_n \in \mathcal{M}$ pairwise disjoint. Then $\forall A \subseteq \mathbb{R}$ one has

$$(*) \quad m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(A \cap E_i).$$

Pf. By MI, we only do the case when $n = 2$.
For this, note that
$m^*(A \cap (E_1 \cup E_2))$

$$= m^*\Big(\underbrace{(A \cap (E_1 \cup_0 E_2)) \cap E_2}_{A \cap E_2}\Big) + m^*\Big(\underbrace{(A \cap (E_1 \cup_0 E_2)) \cap \tilde{E}_2}_{A \cap E_1}\Big) \quad (\because E_2 \in \mathfrak{M}) \quad (\because E_1 \cap E_2 = \emptyset)$$

So (*) is established.

Lemma 2. Let $\{E_n : n \in \mathbb{N}\} \subseteq \mathfrak{M}$ pairwise disjoint.
Then, $\forall A \subseteq \mathbb{R}$

$$m^*\left(A \cap \bigcup_{i=1}^{\infty}{}_0 E_i\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

pf. By $\uparrow$ of m^* and Lemma 1,

$$m^*\left(A \cap \bigcup_{i=1}^{\infty}{}_0 E_i\right) \geq m^*\left(A \cap \bigcup_{i=1}^{n}{}_0 E_i\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

So

$$m^*\left(A \cap \bigcup_{i=1}^{\infty}{}_0 E_i\right) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

(& hence equal as m^* is countably subadditive).

Lemma 3. Let $E := \bigcup_{n=1}^{\infty}{}_0 E_n$, where $\{E_1, E_2, \cdots\}$ is pairwise disjoint family of measurable sets $E_n \in \mathfrak{M}$ $\forall n$. Then $E \in \mathfrak{M}$.

proof. Let $m^*(A) < +\infty$ and. Since $\mathfrak{M}$ is an algebra,

$$m^*(A) = m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) + m^*\left(A \cap \widetilde{\bigcup_{i=1}^{n} E_i}\right)$$

$$\overset{L1}{=} \sum_{i=1}^{n} m^*(A \cap E_i) + \text{''}$$

$$\geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$$

Letting $n \to \infty$,

$$+\infty > m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$$

$$\geq m^*\left(\bigcup_{i=1}^{\infty} (A \cap E_i)\right) + m^*(A \cap \tilde{E}) \quad \left(\text{countable subadd. of } m^*\right)$$

$$= m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) + m^*(A \cap \tilde{E})$$

$$= m^*(A \cap E) + m^*(A \cap \tilde{E}),$$

implying that E is measurable as $A \subseteq \mathbb{R}$ is arbitrary with $m^*(A) < +\infty$ (your Exercise)

Lemma 4. $\mathcal{M}$ is an σ-algebra and $m := m^*\big|_{\mathcal{M}}$ is countably additive (letting $A = \mathbb{R}$ in Lemma 2), so $(\mathcal{M}, m)$ is a measure space (i.e. $\mathcal{M}$ is an σ-alg and m is a measure), and $\mathcal{M}_0, \mathcal{T}, \mathcal{B} \subseteq \mathcal{M}$, generalizing $\ell(I) = m(I)\ \forall I \in \mathcal{I}$

Th (Littlewood 小木人定理): 几乎所有的(可测)集都是好的(指开集等). For any set $E \subseteq \mathbb{R}$, the following statements are equivalent:

(i) $E \in \mathcal{M}$

(ii) Outer-Regularity: $\forall \varepsilon > 0$, $\exists$ open $G \supseteq E$ s.t. $m^*(G \setminus E) < \varepsilon$.

(iii) (also called) Outer-Regularity: $\exists$ G_δ-set $H \supseteq E$ s.t. $m^*(H \setminus E) = 0$

(iv) Inner-Regularity: $\forall \varepsilon > 0$, $\exists$ closed $F \subseteq E$ s.t. $m^*(E \setminus F) < \varepsilon$.

(v) (also called) Inner-regularity: $\exists$ F_σ-set $K \subseteq E$ s.t. $m^*(E \setminus K) = 0$

Moreover, if $m^*(E) < +\infty$ then each of (i)—(v) implies

(vi) $\forall \varepsilon > 0$, $\exists$ $U := \overset{n}{\underset{i=1}{\dot{\bigcup}}} I_i$, $I_1, I_2, \cdots, I_n$ disjoint open intervals 人 s.t. $m(E \triangle U) < \varepsilon$.

Finally (vi) implies each of (i)—(v).

Thus (vi) is a sufficient condition for $E \in \mathfrak{M}$ but not a necessary condition (unless $m(E) < +\infty$).

Proof.

(i) $\Rightarrow$ (ii): As an intermediate step we assume additionally $m^*(E) < +\infty$, so by (i), $m(E) < +\infty$. Let $\varepsilon > 0$. By quasi-outer-regularity $\exists\, G \in \mathcal{T}$ s.t. $G \supseteq E$ and $m^*(G) < m^*(E) + \varepsilon$.

i.e. $m(G) < m(E) + \varepsilon$ (since we now know from I that $G \in \mathfrak{M}$). Hence

$$m(G) - m(E) < \varepsilon$$

$$\begin{array}{c} \shortparallel \\ m(G \setminus E) \end{array} \quad \left(\because m(G) = m(G\setminus E) + m(E) \ \ \& \ \ m(E) < +\infty \right).$$

Thus (i) $\Rightarrow$ (ii) if $m(E) < +\infty$. In general for $E \in \mathfrak{M}$ we have $(\forall n \in \mathbb{N})$ some open

$$G_n \supseteq E \cap [-n, n] \text{ s.t. } m\left(G_n \setminus (E \cap [-n,n])\right) < \frac{\varepsilon}{2^n}$$

Let $G := \bigcup_{n=1}^{\infty} G_n \in \mathcal{T}$, $G \supseteq E$ and

$$G \setminus E \subseteq \bigcup_{n=1}^{\infty} \left(G_n \setminus (E \cap [-n,n]) \right)$$

so $m(G \setminus E) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$.

$\therefore$ (i) $\Rightarrow$ (ii) in general

(ii) $\Rightarrow$ (iii): By (ii), take open $G_n \supseteq E$ s.t. $m^*(G_n \setminus E) < \frac{1}{n}$. Set $H = \bigcap_{n=1}^{\infty} G_n \, (\supseteq E)$.

We have G_δ-set $H \supseteq E$ and

$$m^*(H \setminus E) \leq m^*(G_n \setminus E) < \frac{1}{n} \quad \forall n$$

So $m^*(H \setminus E) = 0$

(iii) $\Rightarrow$ (i): By I, $H \setminus E \in \mathcal{M}$, $H \in \mathcal{M}$ so $E = H \setminus (H \setminus E) \in \mathcal{M}$ as $\mathcal{M}$ is an algebra

Thus (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\quad$ ($m^*(E) = m(E)$ if $E \in \mathcal{M}$)

That (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) then easily follows

e.g. (i) (& (ii)) $\Rightarrow$ (iv): Let $E \in \mathcal{M}$. Then $\widetilde{E} \in \mathcal{M}$ and $\forall \varepsilon > 0$ $\exists$ open $G \supseteq \widetilde{E}$ s.t. $m(G \setminus \widetilde{E}) < \varepsilon$

$$m(G \setminus \widetilde{E}) = m(E \setminus \widetilde{G}) < \varepsilon$$

so (iv) holds with closed $F := \widetilde{G}$.

$\therefore$ (i) – (v) mutually equivalent

(i) $\Rightarrow$ (vi) Under additional assumption

$m^*(E) < +\infty$.

Thus let $m(E) < +\infty$. We wish to establish (vi).

Let $\varepsilon > 0$. By Quasi-Outer-Regularity, $\exists$ open $G \supseteq E$

s.t. $m(G) < m(E) + \varepsilon$. (so $m(G \setminus E) < \varepsilon$)

$$G := \bigcup_{n=1}^{\infty} {}^{\circ} I_n \qquad \left(\text{so } \sum_{n=1}^{\infty} m(I_n) < +\infty \right)$$

and take $N \in \mathbb{N}$ s.t.

$$\sum_{n=N+1}^{\infty} m(I_n) < \varepsilon$$

Take $U := \bigcup_{n=1}^{N} {}^{\circ} I_n$. Then

$$E \triangle U \subseteq (G \setminus E) \cup \left(\bigcup_{n=N+1}^{\infty} {}^{\circ} I_n \right)$$

of measure $< 2\varepsilon$,

showing (vi)

(vi) $\Rightarrow$ (i). Let $\varepsilon > 0$ and $m^*(E \triangle U) < \varepsilon$ where U as in (vi). By Quasi-Outer Regularity, take open $G \supseteq E \triangle U$ ($\supseteq E \setminus U$) with $m(G) < \varepsilon$. Then $W := G \cup U \in \mathcal{T}$ contains E and

$$W \setminus E \subseteq G \cup (U \setminus E) \subseteq G \cup (U \triangle E)$$ of outer-mea. $< \varepsilon + \varepsilon = 2\varepsilon$

Since $\varepsilon > 0$ is arbitrary, (i) holds.

Cor 1. $\mathfrak{M} = \{B \cup Z : B \in \mathcal{B}, \text{ and } Z \in \mathfrak{M}_0\}$
$= \{B \setminus Z : B \in \mathcal{B}, \text{ and } Z \in \mathfrak{M}_0\}$

Pf. We now that $\mathcal{B}, \mathfrak{M}_0 \subseteq \mathfrak{M}$. Conversely let $E \in \mathfrak{M}$. Then $\exists$ G_δ-set $H \supseteq E$ (so $H \in \mathcal{B}$) with that $Z := H \setminus E \in \mathfrak{M}$ so $E = H \setminus Z$. Similarly $\exists$ F_σ-set $K \subseteq E$ s.t. $m(E \setminus K) = 0$ and hence $E = \underbrace{(E \setminus K)}_{\mathfrak{M}_0} \cup \underbrace{K}_{\mathcal{B}}$

Cor 2. Define, $\forall A \subseteq \mathbb{R}$.

$$m_*(A) = \overset{\text{max in } \mathbb{R}^*}{\cancel{\sup}}\{m(K) : F_\sigma\text{-set } K \subseteq A\} \overset{?}{=} m(A) \text{ if } A \in \mathfrak{M}$$
$$= \sup\{m(F) : \text{closed } F \subseteq A\}. \quad (\text{so } m_*(A) \le m^*(A))$$

($m(F) \le m(E)$ if $F \subseteq E \in \mathfrak{M}$ so $\sup \cdots \le m(E)$. Moreover, $\forall \varepsilon > 0$, $m(E) - \varepsilon < m(F)$ for some F $\overset{\wedge}{\sup \ldots}$)

Then $E \in \mathfrak{M} \Rightarrow m_*(E) = m(E)$ and the converse holds provided that $m_*(A) = m^*(A) < +\infty$

Pf. Suppose $m_*(A) = m^*(A) < +\infty$ and $\varepsilon > 0$. Then $\exists F \subseteq A \subseteq G$ with $F \subseteq A \subseteq G$ s.t. with F closed & G open s.t. $m(F) > m_*(A) - \varepsilon$, $m(G) < m^*(A) + \varepsilon$ so $m(G \setminus F) = m(G) - m(F) < 2\varepsilon$. Setting $\varepsilon = \frac{1}{2n}$, we have $F_n \subseteq A \subseteq G_n$ $\forall n$
Setting

$H := \bigcap_{n=1}^{\infty} G_n$ and $K := \bigcup_{n=1}^{\infty} F_n$ we have $K \subseteq A \subseteq H$

and $H \setminus K \in \mathfrak{M}_0$. Hence

$$A = K \cup \underbrace{(A \setminus K)}_{\mathcal{B}} \text{ with } K \in \mathfrak{M} \ \& \ A \setminus K \ (\subseteq H \setminus K) \in \mathfrak{M}_0$$

Counter-example: if $m^*(A) = +\infty = m_*(A)$ but not nece. $A \in \mathfrak{M}$.

e.g. $A = P \cup (-\infty, 0)$ where $P \subseteq (0,1)$ non-measurable

Then $m_*(A) = +\infty = m^*(A)$ while $A \notin \mathfrak{M}$.

r

Ex 1. $m^*(I) = \ell(I)$ $\forall$ interval I.

Sol. Suppose $I = [a,b]\ (\subseteq \mathbb{R})$ special case. Then $I \subseteq \left(a-\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2}\right)$ $(\varepsilon > 0)$ so $m^*(I) \leq (b-a)+\varepsilon$, $\forall \varepsilon > 0$. Hence $m^*(I) \leq \ell(I)$. By the Total length Th, $\forall$ COIC $\{I_n : n \in \mathbb{N}\}$ one has $\sum_{n=1}^{\infty} \ell(I_n) \geq \ell(I)$ so $m^*(I) \geq \ell(I)$. Done this special case. Moreover, $\forall$ finite interval I $(\ell(I) < +\infty)$, one has $m^*(I) = m^*(\bar{I})$ (why?) and $\ell(I) = \ell(\bar{I})$; it follows from our result on the special case that $m^*(I) = \ell(I)$.

Finally consider the case when $\ell(I) = +\infty$. Then, $\forall n \in \mathbb{N}$, $\exists\ I_n \subseteq I$ of length n s.t. $\ell(I_n) = n$. Then

$$m^*(I) \geq m^*(I_n) = \ell(I_n) = n \Rightarrow \infty$$

so $m^*(I) = \infty = \ell(I)$. Done all possible cases for interval I: $m^*(I) = \ell(I)$.

Ex 2. Let $I_i = (a_i, b_i)$ $(i = 1, 2)$ and

$$\bar{z} \in I_1 \cap I_2 .$$

Let $a = \min\{a_1, a_2\}$ and $b = \max\{b_1, b_2\}$. Suppose $a = a_1$ and $b = b_2$. Then

(#) $(a_1, \bar{z}) \subseteq I_1$ and $(\bar{z}, b_2) \subseteq I_2$

and hence that $(a, b) \subseteq I_1 \cup I_2 \subseteq (a, b)$ (*)

(i.e. $(a, b) = I_1 \cup I_2$). Thus $I_1 \cup I_2$ is always an interval if they have a common point.

Sol. By assumptions a is the left-end-point of I_1 & $\bar{z} \in I_1$ so $(a, \bar{z}) \subseteq I_1$ so (#) is clear and hence $(a, b) \subseteq I_1 \cup I_2$ while the 2nd inclusion in (*) is clear by definition of a, b.

The reader should check the last assertion.

Ex 3. Let $\mathcal{C}$ be a collection of open intervals containing a common ξ. Then the union W of members of $\mathcal{C}$ is an interval.

Sol. Let $\xi_1 < x < \xi_2$ with $\xi_1, \xi_2 \in W$.
By ..., it sufficient to show that $x \in W$.
To do this, take $I_1, I_2 \in \mathcal{C}$ s.t. $\xi_1 \in I_1$, $\xi_2 \in I_2$. Then

$$\xi_1, \xi_2 \in I_1 \cup I_2 \text{ while } I_1 \cup I_2$$

is an interval by Ex. as $\xi \in I_1 \cap I_2$.
It follows from $\xi_1 < x < \xi_2$ that $x \in I_1 \cup I_2$ so $x \in W$.

Structure Theorem of Open Sets:
Any open set G in $\mathbb{R}$ can be expressed as a disjoint, countable union of open

intervals.

Proof. Let $x \in G$ and let I_x denote the union of all open sets contained in G containing x. Then I_x is the largest open interval (why?) contained in G and containing x. Then $\forall x, y \in G$,

I_x either coincide with I_y or disjoint (as $I_x \cup I_y$ is also an interval if $I_x \cap I_y \neq \emptyset$).

Thus $\{I_x : x \in G\}$ is a disjoint family with union equal to G, it is countable (not meaning that G is countable).

because from each I_x: you can then have a 1-1 map from this family into $\mathbb{Q}$.

Ex 4. m^* and $\mathcal{M}$ are tran-inv.

Sol. That for m^* was noted

(If $\{I_n : n \in \mathbb{N}\}$ covers A iff

$\{I_n + x : n \in \mathbb{N}\}$ .... $A + x$ $\forall x \in \mathbb{R}$

and $\ell(I_n + x) = \ell(I_n)$.)

Suppose $E \in \mathcal{M}$ & $x \in \mathbb{R}$. Then

$\widetilde{E + x} = \tilde{E} + x$. Hence, $\forall A \subseteq \mathbb{R}$

$$\overset{m^*(A+x)}{\overset{\|}{m^*(A)}} = m^*(A \cap E) + m(A \cap \tilde{E}))$$

$$= m^*((A+x) \cap (E+x)) + m^*((A+x) \cap (\underbrace{\tilde{E} + x}_{\widetilde{E+x}})),$$

Writing A for $A - x$, it follows that

$$m^*(A) = m^*(A \cap (E+x)) + m^*(A \cap (\widetilde{E+x})) \quad \forall A \subseteq \mathbb{R}$$

and so $E + x \in \mathcal{M}$.

Ex 5. $\mathcal{B} + x = \mathcal{B} \quad \forall x \in \mathbb{R}$.

Sol Easy to check that $\mathcal{B} + x$ is an σ-alg containing $\tau (= \tau + x)$ so $\mathcal{B} \subseteq \mathcal{B} + x$ by the smallest property of $\mathcal{B}$. Hence $\mathcal{B} - x \subseteq (\mathcal{B} + x) - x = \mathcal{B}$ $\forall x \in \mathbb{R}$. Replacing $(-x)$ by x, we have $\mathcal{B} + x \subseteq \mathcal{B}$ (so equal).

3.2. Non-measurable sets

Let $m: \mathcal{A} \to [0, +\infty)$ with translation-invariant σ-alg. $\mathcal{A}$ and translation-invariant measure m such that $\mathcal{I} \subseteq \mathcal{A} \subseteq 2^{\mathbb{R}}$ and

$$m(I) = l(I) \quad \forall\, I \in \mathcal{I}.$$

Then $\mathcal{A} \subsetneqq 2^{\mathbb{R}}$: $\exists\, P \subseteq \mathbb{R}$ s.t. $P \notin \mathcal{A}$.

Pf. Defer to the end of our course.